Analysis of Algorithms

Generating Functions

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Generating Functions.

Our goal in the analysis of algorithms is to derive specific expressions for the value of terms in a sequence of quantities $a_0, a_1, a_2, \ldots$ that measure some performance parameter.

Definition

Given a sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ the function

$$\mathbf{A = \sum_{k > 0} a_k z^k}$$

is called the ordinary generating function (OGF) of the sequence.

We use the notation $[z^k]\, A(z)$ to refer to the coefficient a_k.

Theorem (OGF Operations)

If two sequences $a_0, a_1, a_2, \ldots, a_k, \ldots$ and $b_0, b_1, b_2, \ldots, b_k, \ldots$ are represented by the OGFs $A(z) = \sum_{k \geq 0} a_k z^k$ and $B(z) = \sum_{k \geq 0} b_k z^k$, then the operations given on the next slide produce

OGF's that represent the indicated sequences.

In particular:

A (z) is the OGF for a_0 , a_1, a_2, …, a_n , …

B (z) is the OGF for b_0 , b_1, b_2, …, b_n , …

A (z) + B(z) is the OGF for $a_0 + b_0$, $a_1 + b_1$, $a_2 + b_2$, …

z A(z) is the OGF for 0, a_0, a_1, a_2, …, a_k, …

A' (z) is the OGF for a_1, $2a_2$, $3a_3$,…, ka_k, …

A (z) B (z) is the OGF for a_0b_0, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, …

Proof for Addition

$$A(z) + B(z) = \sum_{k \geq 0} a_k z^k + \sum_{k \geq 0} b_k z^k$$

$$A(z) + B(z) = \sum_{k \geq 0} (a_k + b_k) z^k$$

A (z) + B (z) = $a_0 + b_0$, $a_1 + b_1$, $a_2 + b_2$, …

Proof for Convolution

$$A(z)\ B(z) = \sum_{i \geq 0} a_i z^i \sum_{j \geq 0} b_j z^j$$

$$A(z)\ B(z) = \sum_{i,\, j \geq 0} (a_i b_j)\, z^{(i + j)}$$

Let n = i + j then j = n - i

$$A(z)\ B(z) = \sum_{i,\, n - i \geq 0} (a_i b_{n-i})\, z^n$$

$$A(z)\ B(z) = \sum_{n \geq 0} \left(\sum_{0 \geq k \geq n} (a_k b_{n-k})\, z^n \right)$$

$$A(z)\ B(z) = a_0b_0,\ a_0b_1 + a_1b_0,\ a_0b_2 + a_1b_1 + a_2b_0,\ \ldots$$

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Right Shift

$z\,A(z) = \sum a_{n-1} z^n$ $\qquad$ $0, a_0, a_1, a_2, \ldots, a_{n-1}, \ldots$

Left Shift

$\dfrac{A(z) - a_0}{z} = \sum a_{n+1} z^n$ $\qquad$ $a_1, a_2, \ldots, a_{n+1}, \ldots$

Scaling

$A(\lambda z) = \sum \lambda^n a_n z^n$ $\qquad$ $a_0, \lambda^1 a_1, \lambda^2 a_2, \ldots, \lambda^n a_n, \ldots$

Addition

$A(z) + B(z) = \sum (a_n + b_n) z^n$ $\qquad$ $a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n, \ldots$

Generating Function Solution of Recurrences

Lets examine how generating functions can play in the solution of recurrence relations

After a recurrence relationship describing some fundamental property of an algorithm is derived, generating functions can be used to solve the recurrence.

Generating functions provide a mechanical method for solving many recurrence relations.

Given a recurrence describing some sequence $\{a_n\}$, $n \geq 0$ we can develop a solution using the following steps:

1. Multiply both sides of the recurrence by z^n and sum on n.
2. Evaluate the sums to derive an equation satisfied by the OGF
3. Solve the equation to derive an explicit formula for the OGF.
4. Express the OGF as a power series to get expressions for the coefficients (members of the original sequence.)

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Example #1

Solve the recurrence T (n) = 4 T (n - 1) + 2   for $n \geq 1$ with $a_0 = 0$

This recurrence can also be expressed as **$a_n = 4\,a_{n-1} + 2$**

Step 1: Multiply both sides by z^n and sum on n.

$$\sum a_n z^n = 4 \sum a_{n-1} z^n + 2 \sum z^n + 0$$

Step2: Evaluate the sums to derive an equation satisfied by the OGF.

We know that: $A(z) = \sum a_n z^n$

$$zA(z) = \sum a_{n-1} z^n$$

$$\sum_{i=1} z^n = \frac{1}{1-z}$$

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Note:

$$\sum_{k=1}^{\infty} z^k = \sum_{k=1}^{\infty} z\, z^{k-1}$$

$$\sum_{k=1}^{\infty} z\, z^{k-1} = \sum_{k=0}^{\infty} z\, z^k$$

$$\sum_{k=0}^{\infty} z\, z^k = \frac{z}{1 - z}$$

$$\boxed{\sum_{k \geq 1}^{\infty} z^k = \frac{z}{1 - z}}$$

Substitute:

$$\sum a_n z^n = 4 \sum a_{n-1} z^n + 2 \sum z^n + 0$$

$$A(z) = 4 z A(z) + \frac{2z}{1 - z}$$

Step 3: Solve the equation to derive an explicit formula for the OGF.

$$A(z) – 4z A(z) = 2z / (1 – z)$$

$$A(z) (1 – 4z) = 2z / (1 – z)$$

$$A(z) = 2z / ((1 – 4z) (1 – z))$$

Solve for the coefficients C_0 and C_1 (Partial fraction expansion)

$$\frac{2z}{(1 - 4z)(1 - z)} \rightarrow \frac{c_0}{(1 - 4z)} + \frac{c_1}{(1 - z)} \rightarrow \frac{c_0 (1 - z) + c_1 (1 - 4z)}{(1 - 4z)(1 - z)}$$

$2z = C_0 (1 - z) + C_1 (1 - 4z)$

$2z = C_0 - C_0 z + C_1 - 4 C_1 z$

Let z = 1, then

$2 = C_0 - C_0 + C_1 - 4 C_1$

$2 = - 3 C_1$

C0 = 2/3 and **C1 = - 2/3**

$$A(z) = \frac{2/3}{(1 - 4z)} + \frac{-2/3}{(1 - z)}$$

→ The roots are 4z and z

Step 4: Express the OGF as a power series to get expressions for the coefficients. (Member of the original sequence)

$a_n = (2/3) (4^n - 1^n)$ or $\boxed{a_n = (2/3) (4^n - 1)}$

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Homogeneous Recurrences

$a_0 t_n + a_1 t_{n-1} + \ldots + a_k t_{n-k} = 0$

is called a homogeneous linear recurrence with constant coefficients.

Recurrences are classified by the way in which terms are combined, the nature of the coefficients involved, and the number and nature of previous terms used.

Recurrence Type	Typical Example
• First Order	
- Linear	$a_n = a_{n-1} - 1$
- Nonlinear	$a_n = 1 / (1 + a_{n-1})$
• Second Order	
- Linear	$a_n = a_{n-1} + 2 a_{n-2}$
- Nonlinear	$a_n = a_{n-1} * a_{n-2} + \text{sqrt} (a_{n-2})$
- Variable coefficients	$a_n = (n) a_{n-1} + (n – 1) a_{n-2} + 1$
• t th order	$an = f (a_{n-1}, a_{n-2}, \ldots, a_{n-t})$
• full order	$an = n + a_{n-1} + a_{n-2} + \ldots + a_1$
• divide-and-conquer	$a_n = a_{n/2} + a_{n/2} + n$

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- Reconsider the recurrence $\mathbf{a_n = 4\ a_{n-1} + 2}$.

- The recurrence can be rewritten as $\mathbf{a_n - 4\ a_{n-1} - 2 = 0,}$ which is of type homogeneous linear recurrence $\mathbf{a_0\ t_n + a_1\ t_{n-1} + a_2\ t_{n-2} = 0.}$

  Where $a_0 = 1$, $a_1 = -4$, and $a_2 = -2$

- Trying to solve a recurrence of the form $\mathbf{a_0\ t_n + a_1\ t_{n-1} + a_2\ t_{n-2} = 0}$ by intelligence guesswork suggests trying for solutions of the form

  $\mathbf{t_n = x_n}$ where x is an unknown constant.

- Expanding this solution provides
  $\mathbf{a_0\ x_n + a_1\ x_{n-1} + \dots + a_k\ x_{n-k} = 0}$

- If $x = 0$ we have a trivial solution.

- We are looking for the characteristic equation of the recurrence.
  $\mathbf{P(x) = a_0\ x_k + a_1\ x_{k-1} + \dots + a_k}$

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From algebra we learned that this equation must have exactly k roots.

$$\mathbf{P\,(x) = \Uparrow_{i=1}^{k} (x - r_i)}$$

Consider any root r_i of the characteristic polynomial.

$\mathbf{x = r_i}$ is a solution and $\mathbf{x = r_i^{\,n}}$ is also a solution.

$$\mathbf{t\,(n) = \sum_{i=1}^{k} c_i \; r_i^{\,n}}$$

Therefore, reconsider the previous problem:

$A\,(z) = \sum a_n z^n \;=\; (2 / 3)\,\{(1 / (1 – 4z)) \;+\; (-1 / (1 – z))\}$

$\downarrow$ r = 4 $\qquad$ $\downarrow$ r = 1

$a_n = (2 / 3)\,(4^n – 1^n)$ or $\mathbf{a_n = (2 / 3)\,(4^n – 1)}$

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Problem:

Solve the linear recurrence $a_n = a_{n-1} + 1$ for $n \geq 1$ with $a_0 = 0$.

Sometimes we get a better feel for a recurrence when we evaluate a few terms.

n	$a_{n-1} + 1$	a_n
0	0	0
1	0 + 1	1
2	0 + 1 + 1	2
3	0 + 1 + 1 + 1	3
4	0 + 1 + 1 + 1 + 1	4

Therefore $\mathbf{a_n = a_{n-1} + 1}$ has the solution
$\mathbf{a_n = n}$ or $\sum_{1 \leq k \leq n} 1$

$a_n = a_{n-1} + 1$

$$\sum_{n \geq 1} a_n z^n = \sum_{n \geq 1} a_{n-1} z^n + \sum_{n \geq 1} z^n + 0$$

$$\sum a_n z^n = \sum a_{n-1} z^n + \sum z^n + 0$$

$$A(z) = z A(z) + z / (1 - z)$$

$$A(z) - z A(z) = z / (1 - z)$$

$$(1 - z) A(z) = z / (1 - z)$$

$$A(z) = \frac{z}{(1 - z)(1 - z)}$$

$$A(z) = \frac{z}{(1 - z)^2}$$

Note: if root r has multiplicity then $t_n = r^n$, $t_n = nr^n$, $t_n = n^2 r^n$, …, $t_n = n_{m-1} r^n$ are all distinct solutions.

The general solution must be of the form $a_n = c_1 1^n + c_2 n 1^n$ then $c_1 = 0$ and $c_2 = 1$

$$\boxed{a_n = n}$$

Problem:

Solve the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$ for $n > 1$ with $a_0 = 0$, $a_1 = 1$

$$\sum_{n \geq 1} a_n z^n = 5 \sum_{n \geq 1} a_{n-1} z^n - 6 \sum_{n \geq 1} a_{n-2} z^n + a_1 z^1 + 0$$

$$a(z) = 5z\, a(z) - 6 z^2 a(z) + z$$

↑ right shift applied twice

$$a(z) - 5z\, a(z) + 6 z^2 a(z) = z$$

$$(1 - 5z + 6 z^2)\, a(z) = z$$

$$a(z) = \frac{z}{(1 - 5z + 6z^2)} = \frac{c_1}{(1 - 3z)} + \frac{c_2}{(1 - 2z)}$$

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The general solution must be of the form:

$$a_n = C_1 r_1^n + C_2 r_2^n$$

$$a_n = C_1 3^n + C_2 2^n$$

Solve for c_1 and c_2 Let $z = 1$

$$c_1 (1- 2z) + c_2 (1 – 3z) = z$$

$$c_1 - 2c_1 + c_2 – 3 c_2 = 1$$

$$\boxed{- c_1 - 2c_2 = 1} \qquad \boxed{c_1 + c_2 = 0}$$

$$\begin{array}{r} c_1 + c_2 = 0 \\ - c_1 - 2c_2 = 1 \\ \hline \end{array}$$

$c_1 = 1$ $c_2 = -1$

$$\boxed{a_n = 3^n – 2^n}$$

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Problem:

Solve the recurrence $a_n = -2a_{n-1} + 8a_{n-2}$ for $n > 1$ with $a_0 = 0$, $a_1 = 1$

$$\sum_{n \geq 1} a_n z^n = -2 \sum_{n \geq 1} a_{n-1} z^n + 8 \sum_{n \geq 1} a_{n-2} z^n + a_1 z^1 + 0$$

$$a(z) = -2z\, a(z) + 8z^2 a(z) + z$$

$$a(z) + 2z\, a(z) - 8z^2 a(z) = z$$

$$(1 + 2z - 8z^2)\, a(z) = z$$

$$a(z) = \frac{z}{(1 + 2z - 8z^2)} = \frac{c_1}{(1 - 2z)} + \frac{c_2}{(1 + 4z)}$$

The general solution must be of the form:

$$a_n = C_1 r_1^n + C_2 r_2^n$$

$$a_n = C_1 2^n + C_2 (-4)^n$$

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$a_n = C_1 2^n + C_2(-4)^n$

Solve for c_1 and c_2 Let Z = 1

$c_1 (1- 2z) + c_2 (1 + 4z) = z$

$c_1 - 2c_1 + c_2 + 4 c_2 = 1$

$- c_1 + 5c_2 = 1$ $c_1 + c_2 = 0$

$$\begin{array}{r} c_1 + \; c_2 = 0 \\ - c_1 + 5c_2 = 1 \\ \hline \end{array}$$

$c_1 = -1/6$ $c_2 = 1/6$

$$\boxed{a_n = 1/6\ (2^n - (-4)^n)}$$

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Problem:

Solve the recurrence $a_n = a_{n-1} + a_{n-2}$ for $n > 1$ and $a_1 = 1$, $a_0 = 0$

$$\sum_{n \geq 1} a_n z^n = \sum_{n \geq 1} a_{n-1} z^n + \sum_{n \geq 1} a_{n-2} z^n + \sum_{n \geq 1} a_1 z^1 + 0$$

$$a(z) = z\,a(z) + z^2\,a(z) + z$$

$$a(z) - z\,a(z) - z^2\,a(z) = z$$

$$(1 - z - z^2)\,a(z) = z$$

$$a(z) = \frac{z}{1 - z - z^2}$$

Solve for the roots of $(1 – z – z^2)$

$$m_1 = \frac{-b + (b^2 - 4ac)^{1/2}}{2a} \qquad m_2 = \frac{-b - (b^2 - 4ac)^{1/2}}{2a}$$

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$$m_1 = \frac{-(-1) + (1 - 4(1)(-1))^{½}}{2\,(1)}$$

where a = 1, b = -1, c = -1

$$m_1 = \frac{1 + (1 + 4)^{½}}{2\,(1)}$$

$$m_1 = \frac{1 + \sqrt{5}}{2} \qquad m_2 = \frac{1 - \sqrt{5}}{2}$$

$$a(z) = \frac{z}{\left[z - \frac{1 + \sqrt{5}}{2}\right]\left[z - \frac{1 - \sqrt{5}}{2}\right]} = \frac{z}{(z - r_1)(z - r_2)}$$

$$a(z) = \frac{z}{(z - r_1)(z - r_2)}$$

$$a(z) = \frac{c_1}{(z - r_1)} + \frac{c_2}{(z - r_2)}$$

The general solution must be of the form

$$a_n = C_1 r_1^n + C_2 r_2^n$$

Solve for C_1 and C_2

Since $a_0 = 0$ let $n = 0$

Then $a_0 = 0 = C_1 r_1^0 + C_2 r_2^0$

$C_1 + C_2 = 0$

Since $a_1 = 1$ let $n = 1$

Then $a_1 = 1 = C_1 r_1^1 + C_2 r_2^1$

$C_1 r_1^1 + C_2 r_2^1 = 1$

$$c_1 + c_2 = 0$$
$$c_1 r_1 + c_2 r_2 = 1$$

Substitute $c_2 = -c_1$

$$c_1 r_1 - c_1 r_2 = 1$$
$$c_1 (r_1 - r_2) = 1$$

$$c_1 = \frac{1}{(r_1 - r_2)} = \frac{1}{\frac{(1 + \sqrt{5})}{2} - \frac{(1 - \sqrt{5})}{2}} = \frac{1}{\sqrt{5}}$$

$$c_2 = -1 / \sqrt{5}$$

$$a_n = \frac{1}{\sqrt{5}} \left[\left[\frac{1 + \sqrt{5}}{2} \right]^n - \left[\frac{1 - \sqrt{5}}{2} \right]^n \right]$$

which is de Moivre's formula for the Fibonacci sequence.

Solve:

$a_n = 2\, a_{n-1} + a_{n-2} - 2\, a_{n-3}$ for $n > 2$ with $a_0 = 0$ and $a_1 = a_2 = 1$

The recurrence fits the form $A(z) = \sum a_n z^n$

Solution:

$$\sum_{n \geq 3} a_n z^n = 2z \sum_{n \geq 3} a_{n-1} z^{n-1} + z^2 \sum_{n \geq 3} a_{n-2} z^{n-2} - 2 z^3 \sum_{n \geq 3} a_{n-3} z^{n-3} + \sum a_2 z^2 + \sum a_1 z^1 + 0$$

$$\sum a_n z^n = 2z \sum a_{n-1} z^{n-1} + z^2 \sum a_{n-2} z^{n-2} - 2 z^3 \sum a_{n-3} z^{n-3} + \sum z^2 + \sum z^1$$

$$a(z) = 2z\, a(z) + z^2\, a(z) - 2z^3\, a(z) + z^3 / (1 - z) + z / (1 - z)$$

$$(1 - 2z - z^2 + 2z^3)\, a(z) = (z^3 + z^2) / (1 - z)$$

$$(1 - 2z - z^2 + 2z^3)\, a(z) = z - z^2$$

$$a(z) = \frac{z(1 - z)}{(1 - z)(1 + z)(1 - 2z)}$$

$$a(z) = z / ((1+z)(1 - 2z))$$

$$a(z) = \frac{z}{(1-2z)(1+z)}$$

The general solution must be of the form:

$a_n = C_1 r_1^n + C_2 r_2^n$

$\mathbf{a_n = C_2(2)^n + C_1(-1)^n}$

Solve for coefficients

$z = c_0(1+z) + c_1(1-2z)$

$z = c_0 + c_0 z + c_1 - c_1 2z$

Let $z = 0$ then $0 = c_0 + c_1$

Let $z = 1$ then $1 = 2c_0 + c_1 - 2c_1$

$C_0 = 1/3$ and $C_1 = -1/3$

$\mathbf{a_n = 1/3(2^n - (-1)^n)}$

Analysis of Algorithms

Theorem:

If a_n satisfies the recurrence $a_n = x_1\, a_{n-1} + x_2\, a_{n-2} + \ldots + x_t\, a_{n-t}$ for $n \geq t$, then the generating function

$a(z) = f(z) / g(z)$, where the denominator polynomial is $g(z) = 1 - x_1 z - x_2 z^2 - \ldots - x_t z^t$ and the

numerator polynomial is determined by the initial values $a_0, a_1, \ldots, a_{t-1}$

Proof:

The proof follows the general paradigm for solving recurrences described in the introduction of generating functions.

$$a_n = x_1\, a_{n-1} + x_2\, a_{n-2} + \ldots + x_t\, a_{n-t}$$

$$\sum_{n \geq t} a_n z^n = x_1 \sum_{n \geq t} a_{n-1} z^n + x_2 \sum_{n \geq t} a_{n-2} z^n + \ldots + x_t \sum_{n \geq t} a_{n-t} z^n$$

$$a(z) = x_1\, a(z) - x_2\, z^2\, a(z) + \ldots\, x_t\, z^t\, a(z)$$

The left hand side evaluates to a(z) minus the generating polynomial of the initial values, the first sum on the right side evaluates to za(z) minus a polynomial, and so forth.

Thus a(z) satisfies: $a(z) - u_0 = (x_1\, z\, a(z) - u_1(z)) + \ldots + (x_t\, z^t\, a(z) - u_1(z))$ are of degree at most

$t - 1$ with coefficients depending only on the initial values $a_0, a_1, \ldots, a_{t-1}$

Proof (continued)

Solving the equation for a (z) gives the explicit form a (z) = f (z) / g (z) where g (z) has the form announced in the statement and $f(z) \equiv u_0(z) - u_1(z) - \dots - u_t(z)$ depends solely on the initial values of the recurrence and has degree less than t.

$$f(z) = g(z) \sum_{0 \le n \le t} a_n z^n \quad (\text{mod } z^t)$$

Example:

Solve the recurrence $a_n = 2\, a_{n-1} + a_{n-2} - 2\, a_{n-3}$ for $n > 2$ with $a_0 = 0$ and $a_1 = a_2 = 1$

$g(z) = 1 - 2z - z^2 + 2\, z^3$

$g(z) = (1 - z)(1 + z)(1 - 2z)$

Using the initial conditions write:

$f(z) = (z + z^2)(1 - 2z - z^2 + 2\, z^3) \pmod{z^3}$

$f(z) = (2\, z^5 + z^4 - 3\, z^3 - z^2 + z) \pmod{z^3}$

$$\frac{2\, z^5 + z^4 - 3\, z^3 - z^2 + z}{z^3} = 2\, z^2 + z - 3$$ Remainder = $\mathbf{(-z^2 + z)}$

$f(z) = (z - z^2)$

$f(z) = z(1 - z)$

$$a(z) = \frac{f(z)}{g(z)} = \frac{z(1 - z)}{(1 - z)(1 + z)(1 - 2z)} = \frac{z}{(1 - 2z)(1 + z)}$$

$a_n = 1/3\, (2^n - (-1)^n)$

Another Example:

Solve the recurrence $a_n = 5\ a_{n-1} – 6\ a_{n-2}$ for $n > 1$ with $a_1 = 1,\ a_0 = 0$

$g(z) = 1 – 5z + 6z^2$

$g(z) = (1 – 3z)(1 – 2z)$

$f(z) = z\,(1 – 5z + 6z^2)\ \ (\text{mod } z^2)$

$f(z) = (z – 5z^2 + 6z^3)\ \ (\text{mod } z^2)$

$$\frac{6z^3 – 5z^2 + z}{z^2} = 6z - 5 \quad \text{Remainder} \quad z$$

$$a(z) = \frac{f(z)}{g(z)} = \frac{z}{(1 - 3z)(1 - 2z)} = \frac{c_0}{(1 – 3z)} + \frac{c_1}{(1 – 2z)}$$

$z = c_0\,(1 – 2z) + c_1\,(1 – 3z)$

$z = c_0 – 2C_0 + c_1 – 3\,c_1$

$1 = - c_0 - 2c_1$ and $c_0 - c_1 = 0$

Example: (continued)

$$c_0 - c_1 = 0$$
$$-c_0 - 2c_1 = 1$$

$$-3c_1 = 1$$

$c_0 = 1 \quad c_1 = -1$

$$\boxed{a_n = 3^n - 2^n}$$

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Complex Roots:

Solve the recurrence $a_n = 2\ a_{n-1} - a_{n-2} + 2\ a_{n-3}$ for $n > 2$ with $a_0 = 1$, $a_1 = 0$, and $a_2 = -1$

$g(z) = 1 - 2z + z^2 - 2z^3$
$g(z) = (1 + z^2)(1 - 2z)$

$f(z) = (z - z^{2)} (1 + z^2)(1 - 2z) \pmod{z^3}$

$f(z) = (2Z^5 - 3z^4 + 3z^3 - 2z^2 + z) \pmod{z^3}$

$$f(z) = \frac{2Z^5 - 3z^4 + 3z^3 - 2z^2 + z}{z^3} = 2z^2 - 3z + 3 \text{ Remainder } 1 - 2z$$

$$a(z) = \frac{f(z)}{g(z)} = \frac{(1 - 2z)}{(1 + z^2)(1 - 2z)} = \frac{1}{(1 + z^2)}$$

$$a(z) = \frac{1}{(1 + z^2)} = \frac{c_0}{(1 - iz)} + \frac{c_1}{(1 + iz)}$$

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$1 = c_0\ (1 + iz) + c_1\ (1 – iz)$

$1 = c_0 + c_0\ iz + c_1 – c_1 iz$

Let $z = 0$ then $c_0 = - c_1$

Substitute:

$1 = -c_1 - c_1\ iz + c_1 – c_1 iz$

$1 = -2\ c_1\ iz$

$c_1 iz = -1/2$

$c_0 iz = 1/2$

$$a(z) = \frac{1}{2}\left[\frac{1}{(1 – iz)} - \frac{1}{(1 + iz)}\right]$$

$$a_n = ½\ (\ i^n + (-i)^n\ )$$