Analysis of Algorithms
Generating Functions II

Holger Findling

Analysis of Algorithms

Lecture notes are obtained from "Fundamentals of Algorithmics", Gilles Brassard and Paul Bratley.

Inhomogeneous Recurrences.

The solution of a linear recurrence with constant coefficients becomes more difficult to solve when the recurrence is not homogenous, that is when the linear combination is not equal to zero.

In particular, it is no longer true that any linear combination of solutions is a solution.

Consider the following recurrence:

$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \dots a_k t_{n-k} = b^n p(n)$

On the right hand side we have $b^n p(n)$, where b is a constant and p (n) is a polynomial in n of degree d.

Analysis of Algorithms

Solving Inhomogeneous Recurrences.

Consider the recurrence: $t_n - 2\, t_{n-1} = 3^n$ $t_n + (-2)t_{n-1} = b^n\, p\,(n)$

In this case b = 3 and p (n) = 1, a polynomial of degree 0.

We can convert the non-homogeneous recurrence into a homogeneous recurrence as follows:
Multiply the recurrence by 3.
$3\, t_n - 6\, t_{n-1} = 3^{n+1}$

Replace t_n by n-1
$3\, t_{n-1} - 6\, t_{n-2} = 3^n$ $3^{n-1+1} = 3^n$

Subtract:

$$
\begin{array}{l}
t_n - 2\, t_{n-1} \qquad\qquad = 3^n \\
\quad 3\, t_{n-1} - 6\, t_{n-2} = 3^n \\
\hline
t_n - 5\, t_{n-1} + 6\, t_{n-2} = 0
\end{array}
$$

which is a new homogenous recurrence

Solve:

. $a(z) - 5z\, a(z) + 6z^2\, a(z) = z$

$(1 - 5z + 6z^2)\, a(z) = z$

$a(z) = z / ((1 - 2z)(1 - 6z))$

$a_n = c_1\, 2^n + c_2\, 3^n$ The solution must be of this form.

Analysis of Algorithms

It is no longer true that an arbitrary choice of constants c_1 and c_2 produces a solution to the recurrence even when initial conditions are not taken into account.

The basic solutions

$t_n = 2^n$ and $t_n = 3^n$ are **NOT** solutions to the original recurrence $t_n - 2\, t_{n-1} = 3^n$

The general solution to the original recurrence can be determined as a function of t_0 by solving two linear equations in the unknown's c_1 and c_2.

$c_1 + \quad c_2 = t_0$ $n = 0$

$2\, c_1 + 3\, c_2 = 2\, t_0 + 3$ $n = 1$

Constraints on c_1 and c_2, let $n = 1$,

$c_1 2^n + c_2 3^n = 2\, t_{n-1} + 3^n$

$c_1 2^1 + c_2 3^1 = 2\, t_0 + 3^1$

$2\, c_1 + 3\, c_2 = 2\, t_0 + 3$

Solving for c_1 and c_2

$c_2 = t_0 - c_1$

$2\, c_1 + 3\, (t_0 - c_1) = 2\, t_0 + 3$

$-c_1 = -\, t_0 + 3$

$c_1 = t_0 - 3,\ c_2 = 3$

Therefore the general solution is:

$\mathbf{t_n = (t_0 - 3)\, (2^n) + 3\, (3^n)}$ **or** $\mathbf{t_n = (t_0 - 3)\, (2^n) + 3^{n+1}}$

$\mathbf{t_n} \in$ **Theta (3^n)** regardless of the initial conditions.

Another Example

Solve the recurrence $\mathbf{t_n - 2\, t_{n-1} = (n + 5)\, 3^n}$ **Eq#1**

Replace n in the recurrence by n – 1 and then multiply by – 1

$\mathbf{-6\, t_{n-1} + 12\, t_{n-2} = -6\, (n + 4)\, 3^{n-1}}$ **Eq#2**

Replace n in the original recurrence by n – 2 and then multiply by 9

$\mathbf{9\, t_{n-2} - 18\, t_{n-3} = 9\, (n + 3)\, 3^{n-2}}$ **Eq#3**

Add equation 1, 2, and 3 to obtain a homogenous recurrence.

$t_n - 2\, t_{n-1} = (n + 5)\, 3^n$

$-6\, t_{n-1} + 12\, t_{n-2} = -6\, (n + 4)\, 3^{n-1}$

$9\, t_{n-2} - 18\, t_{n-3} = 9\, (n + 3)\, 3^{n-2}$

$t_n - 8\, t_{n-1} + 21\, t_{n-2} - 18\, t_{n-3} = 0$

$(1 - 8z + 21z^2 - 18z^3)\, a(z) = z$ or $a(z) = z / (1 - 2z)\, (1 - 3z)\, (1 - 3z)$

Therefore all solutions are of the form

$\mathbf{t_n = c_1\, 2^n + c_2\, 3^n + c_3\, n3^n}$

Note: We could have written the characteristic polynomial $x^3 - 8x^2 + 21x - 18 = (x - 2)\,(x - 3)^2$

Any choice of values for the constants c_1, c_2, and c_3 provides a solution to the homogeneous recurrence, however the original recurrence imposes restrictions on these constants.

Analysis of Algorithms

$\mathbf{t_n - 2\, t_{n-1} = (n + 5)\, 3^n}$

$T_1 = 2t_0 + 18$ for n = 1

$T_2 = 2t_1 + 63 = 4t_0 + 99$ for n = 2

$\mathbf{t_n = c_1\, 2^n + c_2\, 3^n + c_3\, n3^n}$

$t_0 = c_1 + c_2$ for n = 0

$2\, t_0 + 18 = 2\, c_1 + 3\, c_2 + 3\, c_3$ for n = 1

$4\, t_0 + 99 = 4\, c_1 + 9\, c_2 + 18\, c_3$ for n = 2

---

3 equations 3 unknowns → solve using calculator

$c_1 = t_0 - 9$, $c_2 = 9$, $c_3 = 3$

The general solution is:

$t_n = (t_0 - 9)\, 2^n + (9)\, 3^n + (3)\, n3^n$

$\mathbf{t_n} =$ $\in$ **Theta** $\mathbf{(n3^n)}$ regardless of the initial conditions.

Analysis of Algorithms

Change of Variable

It is sometimes possible to solve more complicated recurrences by making a change of variable.

Consider the divide-and-conquer recurrence **T(n) = 3T(n/2) + n**

To transform the recurrence into a form that we know how to solve, replace n by 2^i.

$T(2^i) = 3\ T\ (2^i / 2) + 2^i$ Note: $2^i / 2 = 2^i\ 2^{-1} = 2^{i-1}$

$T(2^i) = 3\ T\ (2^{i-1}) + 2^i$

We can write this recurrence

$t_i = 3\ t_{i-1} + 2^i$ -------------- Eq#1

This recurrence is in the form of

$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \ldots a_k t_{n-k} = b^n\ p\ (n)$

Convert $t_i = 3\ t_{i-1} + 2^i$ into a homogenous recurrence.

Multiply by 2

$2t_i = 6\ t_{i-1} + 2^{i+1}$

Replace i by i-1

$2t_{i-1} = 6\ t_{i-2} + 2^i$ -------------------- Eq#2

Analysis of Algorithms

$t_i = 3\, t_{i-1} + 2^i$ ------ Equ #1

$2t_{i-1} = 6\, t_{i-2} + 2^i$ ------ Equ #2

Subtract Equation 2 from Equation 1

$t_i - 3\, t_{i-1} = 2^i$

$2\, t_{i-1} - 6\, t_{i-2} = 2^i$

$t_i - 5\, t_{i-1} + 6\, t_{i-2} = 0$

Solve

$a(z) - 5z\, a(z) + 6\, z^2\, a(z) = z$

$a(z) = z / ((1 - 2z)\, (1 + 3z))$

Therefore the solution is in the form

$\mathbf{a_i = c_0\, 3^i + c_1\, 2^i}$

Note:

$T\,(2^i) = T\,(n)$

$n = 2^i$

$i = \log n / \log 2$

$i = \lg n$

Analysis of Algorithms

$T(n) = c_0\, 3^{\lg(n)} + c_1\, 2^{\lg(n)}$

We know that $c_1\, 2^{\lg(n)} = c_1\, n^{\lg(2)}$ and $\log_2(2) = 1$

$T(n) = c_0\, 3^{\lg(n)} + c_1\, n$

$\mathbf{T(n) = c_0\, n^{\lg(3)} + c_1\, n}$

Solve for $c_0 + c_1$

original recurrence: $T(n) = 3T(n/2) + n$

We know that $T(1) = 1 = \mathbf{c_0\, 1^{\lg(3)} + c_1}$ for $n = 1$

$c_0 + c_1 = 1$

$T(2) = 5 = 3\, T(1) + 2 = 3\, c_0 + 2\, c_1$ for $n = 2$

Substitute:

$c_0 = 1 - c_1$ into $3c_0 + 2\, c_1 = 5$

$3\, c_0 + 2\,(1 - c_0) = 5$

$c_0 + 2 = 5$

$c_0 = 3,\ c_1 = -2$

$\mathbf{T(n) = 3\, n^{\lg(3)} - 2\, n}$

Analysis of Algorithms

We are now ready to solve the most important recurrence that we may encounter in the analysis of algorithms.

Consider: $\mathbf{T(n) = lt(n/b) + c\,n^k \quad n > n_0}$

Where:

$n_0 \geq 1$

$l \geq 1$

$b \geq 2$

$k \geq 0$

c is a positive real number

Let

$t_i = T(b^i n_0)$

$T(b^i n_0) = lT(b^{i-1} n_0) + c(b^i n_0)^k$

$t_i = l\, t_{i-1} + c\, n_0^k\, b^{ik}$

$t_i - l\, t_{i-1} = c\, n_0^k\, b^{ik}$ where $p(i) = c\, n_0^k$ and $a = b^k$

So the right hand side is in the form $a^i\,(p^i)$.

Analysis of Algorithms

The characteristic polynomial for t_i - l t_{i-1} = c $n_0^k\ b^{ik}$ must be of the form

$$A(z) = \frac{c_0}{(1 - lz)} + \frac{c_1}{(1 - b^k z)}$$

(root 1 = l^i , root 2 = b^{ik})

$\mathbf{t_i = c_0\ l^i + c_1\ (b^{ik})}$

Solve for i

n/ b = $b^i\ n_0$

b b^i = n/ n_0 or b^{i+1} = n/ n_0

log b^{i+1} = log (n/ n_0)

i log b = log (n/ n_0)

i = log (n/ n_0) / log b

$\mathbf{i = \log_b (n/ n_0)}$

Therefore:

$c_0\ l^i = l^{\log b\ (n/\ n0)}$

$c_0\ l^i = (n/\ n_0)^{\log b\ (l)}$

We can write

$$T\ (n) = \frac{c_0\ \ n^{\log b\ (l)}}{n_0^{\ \log b\ (l)}} + \frac{c_1\ \ n^k}{n_0^{\ k}}$$

$\mathbf{T\ (n) = c_3\ n^{logbl} + c_4\ n^k}$ **-- #1**

Analysis of Algorithms

The initial recurrence $T(n) = lt(n/b) + c\, n^k$ can be written as
$c\, n^k = T(n) - lt(n/b)$ --- substitute #1 into it

$$cn^k = c_3\, n^{\log\_b\, l} + c_4\, n^k - l\,(c_3\,(n/b)^{\log\_b\, l} + c_4\,(n/b)^k)$$

$$cn^k = (1 - l / b^k)\, c_4\, n^k$$

$$c4 = \frac{c}{1 - l/ b^k}$$

$$\mathbf{T(n) = c_3\, n^{\log\_b\, l} + \frac{cn^k}{1 - l / b^k}}$$

Analysis of Algorithms

If **$l < b^k$** then $c_4 > 0$ and $k > \log_b l$.
Therefore the term $c_4 n^k$ dominates the equation.
We conclude that **$T(n) \in$ Theta (n^k | (n/ n_0) is a power of b).**

If **$l > b^k$** then $c_4 < 0$ and $\log_b l > k$.
The fact that c_4 is negative implies that c_3 is positive.
Therefore the term $c_3 n^{\log\_b l}$ dominates.
$t(n) \in$ Theta ($n^{\log\_b l}$)

If **$l = b^k$**, however, we are in trouble because the formula for c_4 involves a division by zero.
In this case the characteristic polynomial has a single root of multiplicity 2 rather than two distinct roots.
Therefore the equation does not provide the general solution to the recurrence.
The general solution in this case would be

$$t_i = c_5 (b^k)l + c_6 \, l \, (b^k)^i$$

$$T(n) = c_7 n^k + c_8 n^k \log\_b (n/ n_0)$$

$T(n)$ ε Theta ($n^k \log_b (n)$)

Analysis of Algorithms

$$T(n)\ \varepsilon \begin{cases} \text{Theta } (n^{k}) & \text{if } l < bk \\ \text{Theta } (n^{k} \log_{b} (n)) & \text{if } l = b^{k} \\ \text{Theta } (n^{\log\_b\ l}) & \text{if } l > b^{k} \end{cases}$$

Analysis of Algorithms

Example #1

Solve the recurrence T (n) = 5 T (n/ 4) + n^2

$$T(n) = c_3\, n^{\log\_b (l)} + \frac{cn^k}{1 - l / b^k}$$

l = 5, b = 4, k = 2

$$T(n) = c_3\, n^{\log\_4(5)} + \frac{cn^2}{1 - 5 / 4^2}$$

if $l < b^k$, $5 < 4^2$

then **T(n) = Theta (n^2)**